Supplemental Information: Distortion of isochronous layers in ice revealed by Ground Penetrating Radar

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The internal horizons shown in Fig. 2b undulate to give arches and troughs which are unrelated to bed topography. The aim is to diagnose whether the arches and troughs were formed by local anomalies in accumulation rate or local anomalies in vertical strain rate. The text shows equations that indicate that the size of features resulting from anomalies in accumulation rate depends almost linearly with depth and they have their maximum growth rate at the surface; whereas the size of features resulting from anomalies in vertical strain rate varies parabolically with depth, and the growth rate is a minimum at the surface. We use this to show which features were formed by each process. This supplemental information shows a more complete derivation of the equations in the text.

Taking the x-axis horizontal and the z-axis vertically downwards and measuring depth in ice equivalent, the burial rate of ice which originally entered the ice sheet surface at the arbitrary point $x = x_0, z = 0$ can be approximately given by

$$\frac{dz}{dt} = w_0 + (x - x_0)\frac{\partial w}{\partial x}|_{x=x_0} + z\frac{\partial w}{\partial z}|_{z=0} \tag{1}$$

where w is the downward velocity of the ice and w_0 the downward velocity at the point of burial; the second term accounts for lateral gradients in the vertical velocity, and the third term the vertical gradients in the vertical velocity.

For a steady-state ice surface profile to prevail, the vertical velocity at the surface must equal the accumulation rate at that point, $w_0 = \dot{a}_0$. Likewise, the gradients $dw/dx|_{x=x_0} = d\dot{a}(x)/dx = \dot{a}'_0$. We assume that the horizontal velocity is constant, $u = u + 0$ so that $x - x_0 = u_0 t$. Finally, $dw/dz|_{x=x_0} = \dot{\epsilon}_0$, the vertical strain rate at that point, negative indicating thinning layers. We assume that this strain rate varies only with x and not depth. Substituting these into Eq. 1 gives:

$$\frac{dz}{dt} = \dot{a}_0 + \dot{a}'_0 u_0 t + \dot{\epsilon} z \tag{2}$$

Eq. 2 has an analytic solution:

$$z(x_0, t) = \frac{\dot{a}_0}{\dot{\epsilon}_0}\left[e^{\dot{\epsilon}_0 t} - 1\right] + \frac{\dot{a}'_0 u_0}{\dot{\epsilon}_0^2}\left[e^{\dot{\epsilon}_0 t} - 1 - \dot{\epsilon}_0 t\right] \tag{3}$$

On adjacent particle paths where the same isochronous layer (constant t) is buried to different depths forming an arch or trough, its size δz can be given as

$$\delta z = z(x_0, t) - z(x_1, t) \tag{4}$$

where the subscripts denote values for ice that began at the two surface positions $x = x_0$ and $x = x_1$, respectively. Substituting into Eq. 3 gives:

$$\delta z = \frac{\dot{a}_0}{\dot{\epsilon}_0}\left[e^{\dot{\epsilon}_0 t} - 1\right] + \frac{\dot{a}_0' u_0}{\dot{\epsilon}_0^2}\left[e^{\dot{\epsilon}_0 t} - 1 - \dot{\epsilon}_0 t\right] - \frac{\dot{a}_1}{\dot{\epsilon}_1}\left[e^{\dot{\epsilon}_1 t} - 1\right] - \frac{\dot{a}_1' u_1}{\dot{\epsilon}_1^2}\left[e^{\dot{\epsilon}_1 t} - 1 - \dot{\epsilon}_1 t\right] \tag{5}$$

Case 1

If the different burial depths result from different accumulation rates ($\dot{a}_0$ and $\dot{a}_1$) while vertical strain rate is constant ($\dot{\epsilon}_0$), then

$$\delta z = (\dot{a}_0 - \dot{a}_1)\left[\frac{e^{\dot{\epsilon}_0 t} - 1}{\dot{\epsilon}_0}\right] + (\dot{a}_0' u_0 - \dot{a}_1' u_1)\left[\frac{e^{\dot{\epsilon}_0 t} - 1 - \dot{\epsilon}_0 t}{\dot{\epsilon}_0^2}\right] \tag{6}$$

Expanding the exponential terms ($e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$) gives

$$\delta z = (\dot{a}_0 - \dot{a}_1)\left[t + \frac{\dot{\epsilon}_0 t^2}{2} + ...\right] + (\dot{a}_0' u_0 - \dot{a}_1' u_1)\left[\frac{t^2}{2} + \frac{\dot{\epsilon}_0 t^3}{6} + ...\right] \tag{7}$$

and ignoring terms higher than t^2,

$$\delta z = (\dot{a}_0 - \dot{a}_1)t + \frac{t^2}{2}\left[(\dot{a}_0 - \dot{a}_1)\dot{\epsilon}_0 + \dot{a}_0' u_0 - \dot{a}_1' u_1)\right] \tag{8}$$

Finally, assuming that x_0 is not in the accumulation gradient (i.e., $\dot{a}_0' = 0$), and substituting for $z \sim \dot{a}_0 t$ gives

$$\delta z \approx \frac{(\dot{a}_0 - \dot{a}_1)}{\dot{a}_0} z + \left[(\dot{a}_0 - \dot{a}_1)\frac{\dot{\epsilon}_0}{\dot{a}_0^2} + \frac{\dot{a}_1' u_1}{\dot{a}_0^2}\right]\frac{z^2}{2} \tag{9}$$

Here the z-term, which dominates the near-surface feature growth, reflects the different downward velocities near the surface at the two locations. The z^2-term arises from two sources. The first contribution results from the deepest and shallowest portions of a layer feeling different vertical velocities. It is -5% of δz at one tenth of the ice thickness. The second z^2-term reflects the horizontal advection of the particles from x_1 at speed u_1 through a zone of laterally varying vertical velocity.

Case 2

If the different burial depths result from local variations in vertical strain rate ($\dot{\epsilon}_0$ and $\dot{\epsilon}_1$) while the accumulation rate is constant ($\dot{a}_0$) and its gradient similarly vanishes (i.e., $dota_0' = 0$) so that

$$\delta z = \frac{\dot{a}_0}{\dot{\epsilon}_0}\left(e^{\dot{\epsilon}_0 t} - 1\right) - \frac{\dot{a}_0}{\dot{\epsilon}_1}\left(e^{\dot{\epsilon}_1 t} - 1\right) \tag{10}$$

Expanding the exponent ($e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$) gives

$$\delta z = \frac{\dot{a}_0}{\dot{\epsilon}_0}\left(1 + \dot{\epsilon}_0 t + \frac{\dot{\epsilon}_0^2 t^2}{2} + \frac{\dot{\epsilon}_0^3 t^3}{6} + ... - 1\right) - \frac{\dot{a}_0}{\dot{\epsilon}_1}\left(1 + \dot{\epsilon}_1 t + \frac{\dot{\epsilon}_1^2 t^2}{2} + \frac{\dot{\epsilon}_1^3 t^3}{6} + ... - 1\right) \tag{11}$$

$$\delta z = \frac{\dot{a}_0 t^2}{2}(\dot{\epsilon}_0 - \dot{\epsilon}_1) + \frac{\dot{a}_0 t^3}{6}(\dot{\epsilon}_0^2 - \dot{\epsilon}_1^2) \tag{12}$$

$$\delta z \approx \frac{1}{2\dot{a}_0}(\dot{\epsilon}_0 - \dot{\epsilon}_1)z^2 ... \left[+\frac{1}{6\dot{a}_0^2}(\dot{\epsilon}_0^2 - \dot{\epsilon}_1^2)z^3\right] \tag{13}$$

or

$$\delta z \approx \frac{1}{2\dot{a}_0}(\dot{\epsilon}_0 - \dot{\epsilon}_1)z^2 ... + \text{higher terms} \tag{14}$$

With a strain rate and strain-rate difference of 10^{-3} a^{-1} and accumulation rate of 0.3 m a^{-1}, the second term is insignificant near the surface, but grows with depth to be around 5% at 50 m and 10% at 100 m.

An important point about the z^2 term in Eq. 9 is that both terms are actually negative and they have the opposite curvature to the z^2 term in Eq. 13. The effect of Eq. 9 is to reduce the rate of growth of the features with depth from an initially linear relation. Thus the growth of features due to accumulation slows with depth, while that due to strain rate accelerates with depth, providing a second diagnostic criteria.

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